The temperature dependence of equilibrium plasma density

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Abstract

Temperature dependence of an electron-nuclear plasma equilibrium density is considered basing on known approaches, which are given in [1]-[2]. It is shown that at a very high temperature, which is characteristic for a star interior, the equilibrium plasma density is almost constant and equals approximately to 10^{25} particles per cm^3. At a relatively low temperature, which is characteristic for star surface, the equilibrium plasma density is in several orders lower and depends on temperature as $T^{3/2}$.

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The electron-nuclear plasma has a high density at high temperature inside stars. It may be supposed that a star exists in a steady state, and the plasma is in a thermodynamic equilibrium. And so it may be expected that an equilibrium plasma should have a density which is determined by its energy minimum.

It will be shown below that an energy minimum of equilibrium plasma exists really. It enables to determinate an equilibrium density of plasma as a function of its temperature. It is important that the density of inner regions of stars is equal to $a_0^{-3} \approx 10^{25}$ particles in cm^3 approximately (a_0 is Bohr radius) and it is in several orders lower for surface regions, where it depends on temperature as $T^{3/2}$.

Let us consider an ensemble consisting of a permanent number of particles N at constant temperature T. The equilibrium state at a minimum of free

energy F will be settled by a ensemble volume V change, i.e. by a change of a particle density n:

$$\left(\frac{\partial F}{\partial n}\right)_{N,T} = 0. \tag{1}$$

The chemical potential describes an assemble equilibrium. The direct interaction between nuclei can be neglected in a dense electrically neutral equilibrium plasma. The plasma energy is determined by electron-electron interaction and electron-nucleus interaction.

The chemical potential of an electron gas at high temperature (in Boltzman approximation) is known [1]:

$$\mu_B = kT\ ln\ \xi_B \tag{2}$$

where

$$\xi = \frac{1}{2}\left[\frac{2\pi h^2}{mkT}\right]^{3/2} n_e. \tag{3}$$

and the electron gas density [1]

$$n_e = \frac{2^{1/2} m^{3/2}}{\pi^2 \hbar^3} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{e^{(\varepsilon-\mu)/kT} + 1}. \tag{4}$$

We will search for the chemical potential of plasma at arbitrary temperature as

$$\mu = kT\ ln\ \xi \tag{5}$$

Using the substitution

$$\zeta = \frac{\varepsilon}{kT}, \tag{6}$$

from (4) we obtain

$$\xi = \xi_B \frac{\pi^{1/2}}{2I_1} \tag{7}$$

where

$$I_1 = \int_0^\infty \frac{\zeta^{1/2} d\zeta}{e^{\zeta} + \xi} \tag{8}$$

Thus

$$\frac{d\xi_B}{d\xi} = \frac{2}{\pi^{1/2}} I_1 \Big(1 - \xi \frac{I_3}{I_1}\Big). \tag{9}$$

where

$$I_3 = \int_0^\infty \frac{\zeta^{1/2} d\zeta}{(e^\zeta + \xi)^2}. \tag{10}$$

The free energy of an electron gas is [1]

$$F_e = \frac{N(2m)^{3/2}(kT)^{5/2}\xi}{3\pi^2\hbar^3 n_e} \int_0^\infty \frac{\zeta^{3/2} d\zeta}{e^\zeta + \xi} = \frac{2I_2}{3I_1} NkT \tag{11}$$

where

$$I_2 = \int_0^\infty \frac{\zeta^{3/2} d\zeta}{e^\zeta + \xi}. \tag{12}$$

At high temperature (in Boltzman approximation) when $\xi = \xi_B \ll 1$ and chemical potential $\mu = \mu_B < 0$ and $|\mu_B| \gg 1$, the free energy of an electron gas may be expanded in series. If we conserve the two first terms of the series, we obtain

$$F_e = F_{ideal} + N \frac{\pi^{3/2} a_o^{3/2} e^3}{4(kT)^{1/2}} n_e \tag{13}$$

Here the first term is the free energy of an ideal gas and the second term is the correction for the identity of electrons. This correction is positive as it takes into account that electrons can not take places which are already occupied by other electrons and it causes an increasing incompressibility of electron gas.

From Eq.(13) we obtain

$$\left(\frac{dF_e}{dn_e}\right)_{N,T} = -\frac{2^{1/2}\pi^2 a_o^{3/2} e^3}{3(kT)^{1/2}} \frac{(I_4 I_1 - I_3 I_2)}{I_1^3(1 - \xi I_3/I_1)} \tag{14}$$

where a_o is the Bohr radius and

$$I_4 = \int_0^\infty \frac{\zeta^{3/2} d\zeta}{(e^\zeta + \xi)^2}. \tag{15}$$

In order to take into account a role of nuclei in the plasma energy formation, we must calculate the correlation correction [2]:

$$\delta F_{corr} = -N \frac{2\pi^{1/2} e^3 kT}{3} \left[Z^2 \frac{n_i}{kT} + \left(\frac{\partial n_e}{\partial \mu} \right)_{N,T} \right]^{3/2} \tag{16}$$

where Z is the nucleus charge, $n_i = n_e/Z$ is the nucleus density.

At high temperature (in Boltzman approximation) this correction is

$$\delta F_{corr} = -N \frac{2\pi^{1/2} e^3 (Z+1)^{3/2}}{3(kT)^{1/2}} n_e^{1/2} \tag{17}$$

This correction takes into account that nuclei "condense" the electron gas in their closeness. On the contrary, the electron gas density decreases away from the nuclei and its compressibility increases. This explains why this correction is negative.

Thus the free energy of plasma at high temperature with account for both corrections

$$F = F_{ideal} + N \frac{\pi^{3/2} a_o^{3/2} e^3}{4(kT)^{1/2}} n_e - N \frac{2\pi^{1/2} e^3 (Z+1)^{3/2}}{3(kT)^{1/2}} n_e^{1/2} \tag{18}$$

i.e. the free energy of plasma has two correction added to ideal value. They have different signs and differently depend on particle density, but they equally depend on temperature. This enables to calculate the equilibrium density of a dense plasma at high temperature from balance condition Eq.(1)

$$n_0 = \frac{16(Z+1)^3}{9\pi^2 a_o^3} \tag{19}$$

In this case the Fermi energy of electron gas $\varepsilon_F \approx \frac{Z^2 e^2}{a_0}$ and the Boltzman approximation is applicable when $kT \gg \varepsilon_F$. In this approximation the equilibrium density does not depend on temperature.

For arbitrary temperature starting from Eq.(16), we obtain

$$\left(\frac{dF_{corr}}{dn_e}\right)_{N,T} = \frac{\pi^{1/2}e^3(Z+1)^{3/2}}{3(kTn_e)^{1/2}}\Big[1-\frac{\xi I_3}{(Z+1)I_1}\Big]^{3/2} *$$
$$*\Big[1-\frac{3\xi}{(Z+1)I_1}\frac{(I_3-2\xi I_5+\xi I_3^2/I_1)}{(1-\xi I_3/I_1)[1-\frac{\xi I_3}{(Z+1)I_1}]}\Big] \qquad (20)$$

The equilibrium density of plasma is

$$\frac{n_e}{n_0} = \Big\{\frac{3}{2^{5/2}\pi^{1/2}}\frac{I_1^3(1-\xi I_3/I_1)}{(I_4I_1-I_2I_3)}\Big[1-\frac{\xi I_3}{(Z+1)I_1}\Big]^{3/2} *$$
$$*\Big[1-\frac{3\xi}{(Z+1)I_1}\frac{(I_3-2\xi I_5+\xi I_3^2/I_1)}{(1-\xi I_3/I_1)[1-\frac{\xi I_3/I_1}{(Z+1)}]}\Big]\Big\}^2 \qquad (21)$$

where

$$I_5 = \int_0^\infty \frac{\zeta^{1/2}d\zeta}{(e^\zeta+\xi)^3}. \qquad (22)$$

At the same time

$$T = \frac{2\pi a_0 e^2}{k}\Big(\frac{n_e}{\pi^{1/2}\xi I_1}\Big)^{2/3}. \qquad (23)$$

The numerically calculated dependence of the equilibrium plasma density on temperature at Z=1 obtained from these equations is shown in Fig.1.

Thus, at a high temperature $T \gg E_F/k$ the equilibrium density of plasma approaches to n_0, and at a relatively low temperature ($T \approx E_F/k$)

$$n_e \sim T^{3/2}. \qquad (24)$$

It is a consequence of that the full energy of electron gas in a general case is [1]

$$E_e = N\frac{2^{1/2}m^{3/2}(kT)^{5/2}\xi}{\pi^2\hbar^3 n_e}\int_0^\infty \frac{\zeta^{3/2}d\zeta}{e^\zeta+\xi} \qquad (25)$$

or with account for above formulas

$$E_e = NkT\frac{I_2}{I_1}. \tag{26}$$

For a degenerate electron gas

$$E_e = \frac{3}{10}NE_F. \tag{27}$$

As in this case the ratio $I_2/I_1 \approx 1$, from Eq.(26) we obtain for the equilibrium degenerate plasma

$$E_F \approx kT. \tag{28}$$

and with account for the definition of E_F, we obtain Eq.(24).

The obtained equilibrium values of a plasma density enables to explain a series of astrophysical effects: the proportionality of star magnetic moments to their angular momenta (the Blackett effect), the star mass spectrum and others [3].

References

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Figure 1: The numerically calculated dependence of the equilibrium plasma density on temperature at Z=1.